Journal of Computational and Applied Mathematics 47 (1993) 61–66
North-Holland

CAM 1264

Dichotomy of a special recurrence relation from the earth sciences

M. van Veldhuizen, A.F. Bervoets
Department of Mathematics and Computer Science, Vrije Universiteit, Amsterdam, Netherlands

W. Zijl
TNO Institute of Applied Geoscience, Delft, Netherlands

Received 15 October 1991

Abstract

Van Veldhuizen, M., A.F. Bervoets and W. Zijl, Dichotomy of a special recurrence relation from the earth sciences, Journal of Computational and Applied Mathematics 47 (1993) 61–66.

The partial differential equation for the groundwater flow in a perfectly layered earth is solved by means of Fourier analysis. The Fourier coefficients in the successive layers are coupled by a recurrence relation with wildly varying coefficients. The dichotomy of this recurrence relation is proved.

Keywords: Recurrence relation; dichotomy.

1. Introduction

In this note we consider the dichotomy of a special matrix-vector recurrence relation in two space dimensions. It is well known that the stable computation of solutions of such recurrence relations depends on the suitable algorithmic use of this dichotomy. See [5,7].

The origin of the recurrence relation considered here is in hydrology and geo-electricity. In order to fully appreciate the meaning of the parameters, we shall briefly describe the origin of the recurrence relation.

Suppose one wants to compute the groundwater movement in a perfectly layered earth. For this note it suffices to consider problems without wells. We use Cartesian x, y, z-coordinates, with the positive z-axis pointing downward. We assume that all layers have constant thickness and permeability. The transport velocity of the groundwater is usually modeled by Darcy's law and the continuity equation, cf. [1]. As a result, the transport velocity is given by the permeability tensor applied to the gradient of a function ϕ, called the *potential*, or *piezometric*

Correspondence to: M. van Veldhuizen, Department of Mathematics and Computer Science, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, Netherlands.

head. In a perfectly layered earth one usually simplifies the permeability tensor to a vertical permeability κ_V, and a horizontal permeability κ_H. The functions κ_H and κ_V are piecewise constant positive functions of z, κ_V and κ_H being constant in each layer. The function ϕ satisfies the differential equation

$$\frac{\partial}{\partial x}\kappa_H(z)\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\kappa_H(z)\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\kappa_V(z)\frac{\partial \phi}{\partial z} = 0, \tag{1.1}$$

The right-hand side vanishes since we consider incompressible flow without wells. At the interface of two layers the interface conditions should hold: the function ϕ and the vertical component of the volumetric flow rate, or flux, $\kappa_V(z)(\partial\phi/\partial z)$ should be continuous. With this proviso we may consider (1.1) in each of the layers separately.

Theoretically one might consider this problem for all x and y, and all $z > 0$. In more practical situations one considers this problem on a domain like

$$\{(x, y, z) \mid z > 0, |x| < L_x, |y| < L_y\}. \tag{1.2}$$

On the vertical sides we impose periodic boundary conditions or vanishing flux. At the (x, y)-plane the piezometric head is given, and at infinite depth the flux should vanish. This problem can be solved by separation of variables. The result is a set of Fourier series, one Fourier series per layer. We mean a Fourier series in the two variables x and y, with z-dependent coefficients.

Consider the wave number vector (ω, σ). The values of ω and σ are related to L_x and L_y respectively, but we do not use this information in this summary. In the jth layer, $z_{j-1} < z < z_j$, $j \geqslant 1$, with depth $d_j = z_j - z_{j-1}$, a special solution of (1.1) is given by

$$\phi_j(x, y, z) = \left(S_j e^{\tau_j(z-z_{j-1})} + D_j e^{-\tau_j(z-z_{j-1})}\right)e^{i\omega x + i\sigma y}, \tag{1.3}$$

where, with function values in the jth layer,

$$\tau_j = \frac{\kappa_H}{\kappa_V}\sqrt{\omega^2 + \sigma^2}. \tag{1.4}$$

The coefficients S_j, D_j may be complex numbers. The interface conditions result in two relations between ϕ_j and ϕ_{j+1} at $z = z_j$. If we denote the vertical permeability in the jth layer by $\bar{\kappa}_j$, then these two relations may be summarized in matrix-vector notation as

$$\begin{pmatrix} \bar{\kappa}_j e^{\tau_j d_j} & -\bar{\kappa}_j e^{-\tau_j d_j} \\ e^{\tau_j d_j} & e^{-\tau_j d_j} \end{pmatrix}\begin{pmatrix} S_j \\ D_j \end{pmatrix} = \begin{pmatrix} \bar{\kappa}_{j+1} & -\bar{\kappa}_{j+1} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} S_{j+1} \\ D_{j+1} \end{pmatrix}. \tag{1.5}$$

Define for $j \geqslant 1$,

$$\mathscr{E}_j = \exp(\epsilon_j), \quad \epsilon_j = \tau_j d_j, \quad f_j = \frac{\bar{\kappa}_j}{\bar{\kappa}_{j+1}} \tag{1.6}$$

and

$$X_j = \begin{pmatrix} S_j \\ D_j \end{pmatrix}, \quad M_j = \frac{1}{2}\begin{pmatrix} (1+f_j)\mathscr{E}_j & (1-f_j)/\mathscr{E}_j \\ (1-f_j)\mathscr{E}_j & (1+f_j)/\mathscr{E}_j \end{pmatrix}. \tag{1.7}$$

Then the recurrence relation (1.5) may be written in matrix-vector notation as

$$X_{j+1} = M_j X_j, \quad j \geqslant 1. \tag{1.8}$$

This is the recurrence relation we want to investigate. The matrix M_j depends on the two parameters f_j and $\mathscr{E}_j$. The parameter f_j is the quotient of two permeabilities. Since permeabilities may vary several orders of magnitude from layer to layer, we can only be sure that $f_j > 0$ for all j. The parameter $\mathscr{E}_j$ depends on the thickness of the jth layer, and on τ. We can only be sure $\mathscr{E}_j > 1$. The parameters $\mathscr{E}_j > 1$ and $f_j > 0$ may vary strongly with index j.

A similar recurrence relation occurs in geo-electrical problems for a perfectly layered earth, cf. [6].

By a solution of the recurrence relation (1.8) we mean an infinite sequence of vectors $\{X_j\}_{j \geqslant 1}$ satisfying (1.8) for all $j \geqslant 1$. A decreasing (increasing) scalar sequence is a sequence of which the successive terms decrease (increase) in modulus.

2. Analysis of the recurrence relation

In this section we shall prove the existence of a one-sided dichotomy for the recurrence relation $X_{j+1} = M_j X_j$. A first observation: the recurrence relation (1.8) is not "slowly varying" in the sense of [8]. To see this, it suffices to consider two limiting cases. First, for very large values of f_j the matrix of eigenvectors of M_j tends to

$$\begin{pmatrix} 1/\mathscr{E}_j & 1 \\ \mathscr{E}_j & -1 \end{pmatrix},$$

and for $f_j \to 0$ the matrix of eigenvectors tends to

$$\begin{pmatrix} 1/\mathscr{E}_j & 1 \\ -\mathscr{E}_j & 1 \end{pmatrix}.$$

In each instance the first eigenvalue is the smaller one. Clearly, the invariant space corresponding to the smaller eigenvalue for large f_j corresponds more or less with the invariant space for the larger eigenvalue for small f_j. That is, the transformations reducing M_j to diagonal form may vary wildly with j, and the recursion is not slowly varying in the sense of [8].

The basis of the analysis is the following simple observation. Let

$$F_j = \frac{1}{2}\begin{pmatrix} 1+f_j & 1-f_j \\ 1-f_j & 1+f_j \end{pmatrix}, \qquad Q = \tfrac{1}{2}\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.1}$$

The symmetric matrix F_j has two positive real eigenvalues 1 and f_j, and the orthogonal matrix Q of eigenvectors of F_j is independent of f_j. Put

$$E_j = \operatorname{diag}\{\mathscr{E}_j, 1/\mathscr{E}_j\}, \qquad D_j = \operatorname{diag}\{1, f_j\}. \tag{2.2}$$

Then, cf. (1.8), $M_j = F_j E_j = Q^{\mathrm{T}} D_j Q E_j$. Put

$$H_j = Q^{\mathrm{T}} E_j Q = \frac{1}{2}\begin{pmatrix} \mathscr{E}_j + 1/\mathscr{E}_j & \mathscr{E}_j - 1/\mathscr{E}_j \\ \mathscr{E}_j - 1/\mathscr{E}_j & \mathscr{E}_j + 1/\mathscr{E}_j \end{pmatrix}. \tag{2.3}$$

Then $M_j = Q^T D_j H_j Q$. Note that D_j, H_j and M_j are invertible. The investigation of the recurrence relation (1.8) amounts to the investigation of a product like $M_p M_{p-1} \cdots M_q$, $p > q$. Apart from a change of coordinates by an orthogonal matrix, this product equals the product $D_p H_p D_{p-1} H_{p-1} \cdots D_q H_q$. This product, or its inverse, is the subject of our investigation. We start with a lemma.

Lemma 2.1. *Assume $f_j > 0$, $\mathscr{E}_j > 1$. Let $\rho \geqslant 0$ be an arbitrary positive real number. Then there exist real numbers $g_{1j} > 1$, $g_{2j} > 1$ and $\rho_1 > 0$, $\rho_2 > 0$ such that*

$$D_j H_j \begin{pmatrix} 1 \\ \rho \end{pmatrix} = g_{1j} \begin{pmatrix} 1 \\ \rho_1 \end{pmatrix}, \qquad H_j^{-1} D_j^{-1} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} = g_{2j} \begin{pmatrix} 1 \\ -\rho_2 \end{pmatrix}.$$

Proof. Let $\nu = (1, \rho)^T$. By direct computation,

$$D_j H_j \nu = \frac{1}{2} \begin{pmatrix} (1+\rho)\mathscr{E}_j + (1-\rho)/\mathscr{E}_j \\ f_j(1+\rho)\mathscr{E}_j + f_j(\rho-1)/\mathscr{E}_j \end{pmatrix} = g_{1j} \begin{pmatrix} 1 \\ \rho_1 \end{pmatrix}.$$

Thus $g_{1j} = \frac{1}{2}(1+\rho)\mathscr{E}_j + \frac{1}{2}(1-\rho)/\mathscr{E}_j$. Recall $\mathscr{E}_j = \exp(\epsilon_j)$, cf. (1.6). Since $\rho \geqslant 0$, we have

$$g_{1j} = \cosh(\epsilon_j) + \rho \sinh(\epsilon_j) > 1. \tag{2.4a}$$

The scalar ρ_1 is defined as the quotient of the second coordinate of $D_j H_j \nu$ and g_{1j}. Clearly,

$$\rho_1 = f_j \frac{\mathscr{E}_j^2 + r(\rho)}{\mathscr{E}_j^2 - r(\rho)}, \quad r(\rho) = \frac{\rho - 1}{\rho + 1}. \tag{2.4b}$$

Observe $r(\rho) \in [-1, 1)$. Since $f_j > 0$ and $\mathscr{E}_j > 1$, we find $\rho_1 > 0$. This proves the first statement. The proof of the second statement is very similar. One easily verifies

$$g_{2j} = \cosh(\epsilon_j) + \frac{\rho}{f_j} \sinh(\epsilon_j) \tag{2.5a}$$

and

$$\rho_2 = \frac{f_j \sinh(\epsilon_j) + \rho \cosh(\epsilon_j)}{\rho \sinh(\epsilon_j) + f_j \cosh(\epsilon_j)}. \qquad \square \tag{2.5b}$$

In view of this result we shall consider solutions of the recurrence relation (1.8) on the basis given by the columns of Q^T. We shall say that a solution of (1.8) is *decreasing* (*increasing*) if its first coordinate on the basis given by the columns of Q^T forms a decreasing (increasing) scalar sequence. Thus, for a decreasing solution $\{X_j\}_{j \geqslant 1}$ the scalar sequence $\{\| X_j \|\}_{j \geqslant 1}$ is not necessarily decreasing.

The following theorem needs stronger assumptions about ϵ_j and f_j. These assumptions are satisfied if the thickness of the layers and the permeabilities are both bounded and bounded away from 0 uniformly in j. In practice, with a finite number of layers, this kind of assumption is not unrealistic. The proof of the theorem is in the spirit of the theory in [8, Section 9].

Theorem 2.2. *Suppose*

(i) $\epsilon_j \geqslant \underline{\epsilon} > 0$ *for all* $j > 0$;

(ii) $\max\{f_j, 1/f_j\} \leqslant \bar{f}$ *for all* $j > 0$.

Then the linear space R of solutions of the recurrence relation (1.8) *contains a one-dimensional linear subspace R_{d} of decreasing solutions $\{X_j\}_{j \geqslant 1}$, which tend to* 0 *for $j \to \infty$. Any solution not belonging to R_{d} is increasing.*

Proof. In this proof we consider the recursion (1.8) on the basis given by the columns of Q^{T}. That is, replace M_j by $D_j H_j$ and X_j by $Q^{\mathrm{T}} z_j$. Construct a solution $\{z_j^{(k)}\}_{j \geqslant 1}$ such that $z_k^{(k)}$ is a scalar multiple of the unit vector $(1, 0)^{\mathrm{T}}$, and such that $z_1^{(k)} = (1, -\rho_k)^{\mathrm{T}}$, for some nonnegative real number ρ_k. This solution exists in view of Lemma 2.1: just start at $j = k$, go backwards to $j = 1$, and perform a proper scaling. Formula (2.5b) and the assumptions (i), (ii) imply the existence of two finite strictly positive real numbers $\bar{\rho}$ and $\underline{\rho}$, such that $\rho_k \in [\underline{\rho}, \bar{\rho}]$. The set of ρ_k is an infinite subset of this segment. Hence, by the Heine–Borel Theorem, the segment contains a limit point $\rho_{\lim}$ of the infinite set of ρ_k. Hence, a subsequence of the ρ_k converges to $\rho_{\lim}$. In view of Lemma 2.1 the first k coordinates of $\{z_j^{(k)}\}$ form a decreasing sequence. Clearly, the first coordinates of the solution starting at $(1, -\rho_{\lim})$ decrease for all $j > 0$.

We now prove the uniqueness of the limit point $\rho_{\lim}$. This uniqueness implies that the set of decreasing solutions is a linear subspace. Suppose $\tilde{\rho}$ is a limit point different from $\rho_{\lim}$. Construct a sequence $\{\nu_j\}_{j \geqslant 1}$ by starting (1.8) with the difference of $(1, -\rho_{\lim})^{\mathrm{T}}$ and $(1, -\rho')^{\mathrm{T}}$. By direct computation we find that the two coordinates of ν_2 have equal sign. Thus Lemma 2.1 applies, showing an increasing sequence of first coordinates. On the other hand, $\{\nu_k\}_{k \geqslant 1}$ is the difference of two sequences with decreasing first coordinate. Note that the rate of growth is at least geometrical, and so is the rate of decrease. Hence, we have reached a contradiction. This proves the uniqueness of the limit point $\rho_{\lim}$.

Finally, since $(1, -\rho_{\lim})^{\mathrm{T}}$ and $(1, \rho_{\lim})^{\mathrm{T}}$ form a basis in two-dimensional space, any solution not starting in a scalar multiple of $(1, -\rho_{\lim})^{\mathrm{T}}$ contains a component with starting vector $(1, \rho_{\lim})^{\mathrm{T}}$. This component is increasing, which proves the last claim. □

Theorem 2.2 shows the one-sided dichotomy in the sense of [4]: any projection on the vector $(1, -\rho_{\lim})^{\mathrm{T}}$ will do. In addition, the growth factors g_{1j}, g_{2j} are uniformly bounded away from 1, corresponding to an exponential dichotomy.

3. Application

We apply these results to the hydrological problem sketched in Section 1. Typically, the number of layers is finite, and the last layer has infinite thickness. In order to bring this in the framework of Theorem 2.2, we subdivide the last layer into an infinite number of layers, all of them with the permeability of the infinite layer. Thus $f_j = 1$ for the interfaces between these artificial layers, and the corresponding matrices M_j are diagonal matrices. The thickness of the artificial layers plays a part in the decreasing solution of the recurrence relation (1.8), but not in the corresponding solution of the differential equation (1.1). The piezometric head at the earth surface is known. That is, the value of $S_1 + D_1$ is known. At infinite depth the gradient of the solution of the differential equation (1.1) should vanish. This means that we need a decreasing solution in the sense of Theorem 2.2. One obtains this solution by Miller's algorithm as follows. In the artificial layers the decreasing solution at index N is a multiple of $e_2 = (0, 1)^{\mathrm{T}}$. Choose

$X_N = e_2$. Compute X_{N-1}, X_{N-2}, up to X_1. Then, in view of Lemma 2.1, and taking into account the basis used in Lemma 2.1, there is a positive ρ such that

$$X_1 = \prod_{j=1}^{N} g_{2j} Q^{\mathrm{T}} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} = \tfrac{1}{2}\sqrt{2} \prod_{j=2}^{N} g_{2j} \begin{pmatrix} 1-\rho \\ 1+\rho \end{pmatrix}.$$

Since $S_1 + D_1$ is known, we must scale the solution $\{X_j\}_{1 \leqslant j \leqslant N}$ by the sum of the coordinates of X_1. Clearly, this sum equals $\sqrt{2}\prod g_{2j} > 1$. This shows that Miller's algorithm is well-defined in this instance.

The algorithm has been implemented, and tested for up to fifty layers with strongly varying permeabilities and geometries. The computed recursion vectors are very accurate. This implementation of the recurrence relation is used as a black box in the program FPFPA (Fourier Potential Flow Pattern Analyser), in use at the TNO Institute of Applied Geoscience, cf. [2,3].

References

[1] J. Bear and A. Verruijt, *Modeling Groundwater Flow and Pollution* (Reidel, Dordrecht, 1987).

[2] A.F. Bervoets, Flow pattern analysis for a potential defined by a Fourier series, TNO report OS-91-30A, Delft, 1991.

[3] A.F. Bervoets, W. Zijl and M. van Veldhuizen, Spatial scale analysis of groundwater flow pattern for transport in sedimentary basins, in: *Proc. European Simulation Symp. on Modelling and Control of Water Resources Systems and Global Changes*, Ghent, 1991.

[4] W.A. Coppel, *Dichotomies in Stability Theory*, Lecture Notes in Math. **629** (Springer, New York, 1978).

[5] W. Gautschi, Computational aspects of three term recurrence relations, *SIAM Rev.* **9** (1967) 24–82.

[6] O. Koefoed, *Geosounding Principles 1, Resistivity Sounding Measurements*, Methods Geochem. Geophys. **14A** (Elsevier, Amsterdam, 1979).

[7] R.M.M. Mattheij, Decoupling and stability of algorithms for boundary value problems, *SIAM Rev.* **27** (1985) 1–44.

[8] A. van der Sluis, Estimating the solutions of slowly varying recursions, *SIAM J. Math. Anal.* **7** (5) (1976) 662–695.